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Saturation Problem of L^p-Approximation by Hermite–Fejér Interpolation

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The saturation of L^p-approximation of Hermite–Fejér interpolation based on the zeros of generalized Jacobi polynomials is considered. Although mean convergence may improve the approximation order compared to uniform convergence, surprisingly, their saturation orders are exactly same, that is, $1/n$. An inverse theorem is also given with respect to L^p-approximation of Hermite–Fejér interpolation.

1. INTRODUCTION

The main purpose of this paper is to investigate the saturation problem of weighted L^p convergence of Hermite–Fejér interpolation based on the zeros of generalized Jacobi polynomials and the inverse problem as well. Let

$$-1 < x_n < x_{n-1} < \cdots < x_1 < 1$$

be the zeros of the generalized Jacobi polynomials which are orthogonal with respect to a generalized Jacobi weight $w := g(x)\, u_1(x)$, $g^{\pm 1} \in L^\infty$ and $u_1(x)$ a Jacobi weight $u := (1-x)^\alpha (1+x)^\beta$ $(-1 < \alpha,\ \beta)$. Given a positive integer n and a function f, the Hermite–Fejér interpolating polynomial $H_n(f, w, x)$ is defined to be the unique polynomial of degree at most $2n-1$ satisfying

$$H_n(f, w, x_k) = f(x_k), \qquad k = 1, ..., n,$$

and

$$H_n'(f, w, x_k) = 0, \qquad k = 1, ..., n.$$

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In 1916, L. Fejér (cf. [10, (5.8), p. 166]) proved Weierstrass' theorem using the above interpolation in which w is taken the Chebyshev weight $v := 1/\sqrt{1-x^2}$, that is

$$\lim_{n \to \infty} \|H_n(f, v, x) - f(x)\| = 0, \tag{1.1}$$

for all $f \in C[-1, 1]$, where $\|\cdot\|$ denotes maximum norm on $[-1, 1]$.

On the other hand, we know that there exists some Hermite–Fejér interpolation such that

$$\lim_{n \to \infty} \|H_n(f, w, x) - f(x)\| = 0, \tag{1.2}$$

does not hold for all $f \in C[-1, 1]$ (cf. [10, Corollary 5.3, p. 174], [12, (14.6.17), p. 343]). This leads to consider its mean convergence. Recently, there have been a lot of papers considering mean convergence of Hermite–Fejér interpolation and Hermite interpolation. For more details, see the survey paper [11] and references therein. We would like to mention that P. Nevai and P. Vértesi [6, Theorem 5, p. 55] originally gave necessary and sufficient conditions for Hermite–Fejér interpolation for all continuous functions. It has been shown that mean convergence may improve the convergence behavior of Hermite–Fejér interpolation compared to the uniform metric.

Considering the degree of approximation, A. K. Varma and J. Prasad [13, Theorem 1, p. 226] obtained that

$$\|H_n(f, v, x) - f(x)\|_{v,p} \leqslant c\omega(f, 1/n) \tag{1.3}$$

for $f \in C[-1, 1]$ and $1 \leqslant p < \infty$, where

$$\|f\|_{u,p} := \left(\int_{-1}^{1} u(x)\, |f(x)|^p\, dx\right)^{1/p}$$

$\omega(f, \cdot)$ the modulus of continuity and the symbol "c" denotes some constant which is positive and independent of the variables and indices. (1.3) implies that

$$\|H_n(f, v, x) - f(x)\|_{v,p} \leqslant \frac{c}{n} \tag{1.4}$$

for $f \in \text{Lip}\, 1$.

It should be mentioned (cf. [11, (3.10), p. 14] and references therein) that $\omega(f, 1/n)$ in (1.3) can be replaced by the Ditzian–Totik modulus of continuity, that is,

$$\|H_n(f, v, x) - f(x)\|_{v,p} \leqslant c\omega_\varphi(f, 1/n), \tag{1.5}$$

where

$$\omega_\varphi(f, t) := \sup_{0<h\leqslant t} \left\| f\left(x + \frac{h}{2}\varphi(x)\right) - f\left(x - \frac{h}{2}\varphi(x)\right) \right\|,$$

the *Ditzian–Totik modulus*.

On the other hand, if $f \in \mathrm{Lip}\, 1$, then

$$\|H_n(f, v, x) - f(x)\| = O\left(\frac{\ln n}{n}\right), \tag{1.6}$$

and the order cannot be improved (cf. [1, Theorem 1, p. 77] or [11, Theorem 5.1, p. 168]). Thus, mean convergence may improve the approximation order compared to the uniform metric. Moreover, we have recently shown [4, Theorem 1, p. 267] that even if f is a polynomial, the corresponding asymptotic rate of $H_n(f, v, x)$ is just $1/n$ with respect to weighted L^2 convergence.

This leads us to raise the following questions:

(1) Does

$$\|H_n(f, w, x) - f(x)\|_{u,p} = o(n^{-1}) \qquad (n \to \infty) \tag{1.7}$$

hold? This is related to the so-called saturation problem.

(2) May we deduce function properties from the L^p-approximation order with respect to Hermite–Fejér interpolation? This is related to the inverse problem of approximation.

This paper will consider the above problems. In order to state the corresponding results, we also introduce the notations.

$$\|f\|_{u,p} := \left(\int_{-1}^{1} u(x)\, |f(x)|^p\, dx\right)^{1/p}$$

and

$$\|f\|_{L_p(D)} := \left(\int_D |f(x)|^p\, dx\right)^{1/p}$$

and

$$\|f\|_{L_u^p(D)} := \left(\int_D u(x)\, |f(x)|^p \, dx \right)^{1/p}$$

where $D \subset (-1, 1)$. Now,

$$\begin{aligned} \Omega_\varphi(f, t)_{u,p} &:= \sup_{<h \leqslant t} \|\Delta_{h\varphi} f\|_{L_u^p(I_h)} \\ &= \sup_{<h \leqslant t} \left\| f\left(x + \frac{h}{2} \varphi(x) \right) - f(x - \frac{h}{2} \varphi(x) \right) \right\|_{L_u^p(I_h)}, \end{aligned}$$

the *Ditzian–Totik weighted main-part modulus*. If $x \pm h\varphi(x) \notin I_h$, the expression inside $\|\cdot\|_{L_u^p(I_h)}$ is taken to be zero, where $\varphi(x) := \sqrt{1-x^2}$ and $I_h := [-1 + 2h^2, 1 - 2h^2]$.

The symbol "$\sim$" is used as follows: if A and B are two expressions depending on some variabled and indices, then

$$A \sim B \Leftrightarrow |AB^{-1}| \leqslant c \qquad \text{and} \qquad |A^{-1}B| \leqslant c.$$

Now we state our results.

THEOREM 1.1 *Let $f \in C^1[-1, 1]$, $w(x)$ be a generalized Jacobi weight, $u(x)$ be a Jacobi weight, and let $1 \leqslant p < \infty$. Then*

$$\|H_n(f, w, x) - f(x)\|_{u,p} = o(n^{-1}) \tag{1.8}$$

if and only if $f \equiv$ constant.

THEOREM 1.2 *Let $f \in C[-1, 1]$ and $1 \leqslant p < \infty$. Then*

$$\|H_n(f, w, x) - f(x)\|_{u,p} = O(n^{-\gamma}) \qquad (0 < \gamma < 1) \tag{1.9}$$

implies that $\Omega_\varphi(f, t)_{u,p} = O(t^\gamma)$, and

$$\|H_n(f, w, x) - f(x)\|_{u,p} = O(n^{-1}) \tag{1.10}$$

implies that $\Omega_\varphi(f, t)_{u,p} = O(t\, |\ln t|)$.

2. PROOFS

First we prove a lemma.

LEMMA 2.1 *If (1.8) is satisfied and $1 \leqslant p < \infty$. Then*

$$\|\sqrt{1-x^2}\, H_n'(f, w, x)\|_{u,p} = o(1). \tag{2.1}$$

Proof. For l given by $l := \max\{k: 2^k \leqslant n\}$, we expand $H_n(f, w, x)$ by

$$H_n(f, w, x) - H_1(f, w, x) = (H_n(f, w, x) - H_{2^l}(f, w, x)) + (H_{2^l}(f, w, x) - H_{2^{l-1}}(f, w, x)) + \cdots + (H_2(f, w, x) - H_1(f, w, x)).$$

If $m < n$, from (1.8) we have

$$\|H_n(f, w, x) - H_m(f, w, x)\|_{u,p} = o\left(\frac{1}{m}\right).$$

Note that for a polynomial $p_n(x)$ of degree $\leqslant n$ we have the following Bernstein–Markov inequality (cf. [3, Theorem 1, p. 1478]):

$$\|(\sqrt{1-x^2})^r p_n^{(r)}(x)\|_{u,p} \leqslant cn^r \|p_n(x)\|_{u,p}. \tag{2.2}$$

Now, let $r = 2$ and applying (2.2) we obtain

$$\|(1-x^2) H_n''(f, w, x)\|_{u,p} \leqslant \sum_{k=1}^{l} 2^{2k} o\left(\frac{1}{2^k}\right) = o\left(\sum_{k=0}^{l} 2^k\right) = o(n). \tag{2.3}$$

On the other hand, we have (cf. [5, Theorem 6.3.14, p. 113]) that

$$\|p_n(x)\|_{u,p} \leqslant c \|p_n(x)\|_{L_u^p(D_n)}, \tag{2.4}$$

where $D_n := [-1 + cn^{-2}, 1 - cn^{-2}]$.

Let $x_0 := 1$, $x_{n+1} := -1$, $x := \cos\theta$ and $x_k := \cos\theta_k$. Note that (cf. [6, Lemma 2, p. 35]) $|\theta_k - \theta_{k+1}| \leqslant c/n$ $(k = 0, 1, ..., n)$, hence, $x_1 - 1 \sim n^{-2}$ and $x_n + 1 \sim n^{-2} (n \to \infty)$, so that $[x_n, x_1] \sim D_n$. Therefore, if $x \in [x_{k+1}, x_k]$, we have

$$\frac{\sin(\theta + \theta_k)/2}{\sin\theta} \sim 1,$$

so there exists $N > 0$ such that

$$\left|\frac{x - x_k}{\sqrt{1-x^2}}\right| = 2\left|\frac{\sin(\theta+\theta_k)/2 \sin(\theta-\theta_k)/2}{\sin\theta}\right| \leqslant \frac{c}{n}$$

for $n \geqslant N$ and $x \in D_n$, which guarantees $x \pm c\varphi(x)/2n \in [-1 + cn^{-2}, 1 - cn^{-2}]$. Note that $H_n'(f, w, x_k) = 0$ $(k = 1, ..., n)$. Hence from (2.4) we have

$$\begin{aligned}
\|\sqrt{1-x^2}\, H_n'(f, w, x)\|_{u,p} &\leqslant c\, \|\sqrt{1-x^2}\, H_n'(f, w, x)\|_{L_u^p(D_n)} \\
&= c\left(\sum_{k=1}^{n-1} \int_{x_{k+1}}^{x_k} u(x) \left|\sqrt{1-x^2} \int_x^{x_k} H_n''(f, w, t)\, dt\right|^p dx\right)^{1/p} \\
&\leqslant c\left(\sum_{k=1}^{n-1} \int_{x_{k+1}}^{x_k} u(x) \left|\sqrt{1-x^2} \int_{x-c\varphi(x)/2n}^{x+-c\varphi(x)/2n} |H_n''(f, w, t)|\, dt\right|^p dx\right)^{1/p} \\
&\leqslant c\left\| u^{1/p}(x)\sqrt{1-x^2} \int_{x-c\varphi(x)/2n}^{x+c\varphi(x)/2n} |H_n''(f, w, t)|\, dt \right\|_{L_p(D_n)} := I_p. \qquad (2.5)
\end{aligned}$$

For $p>1$, the maximal function $M(g)$ satisfies $\|M(g)\|_p \leqslant c_p \|g\|_p$ (cf. [7, Theorem 1, p. 5]) and $u^{1/p}(x)\sqrt{1-x^2} \sim u^{1/p}(t)\sqrt{1-t^2}$ if $t \in [x-c\varphi(x)/2n, x+c\varphi(x)/2n]$. Thus by (2.3) we obtain

$$\begin{aligned}
I_p &\leqslant \frac{c}{n}\left\| u^{1/p}(x)\,(1-x^2)\, \frac{1}{\varphi(x)/2n} \int_{x-c\varphi(x)/2n}^{x+c\varphi(x)/2n} |H_n''(f, t)|\, dt \right\|_{L_p(D_n)} \\
&\leqslant \frac{c}{n}\left\| \frac{1}{\varphi(x)/2n} \int_{x-c\varphi(x)/2n}^{x+c\varphi(x)/2n} u^{1/p}(t)\,(1-t^2)\, |H_n''(f, t)|\, dt \right\|_{L_p(D_n)} \\
&\leqslant \frac{c}{n}\, \|M(u^{1/p}(t)\,(1-t^2)\, H_n''(f, t), x)\|_{L_p(D_n)} \leqslant \frac{c}{n}\, \|(1-x^2) H_n''(f, x)\|_{L_u^p(D_n)} \\
&\leqslant \frac{c}{n}\, \|(1-x^2) H_n''(f, x)\|_{u,p} = o(n)\, \frac{c}{n} = o(1). \qquad (2.6)
\end{aligned}$$

For $p=1$, we have

$$\begin{aligned}
I_1 &= \int_{D_n} u(x)\sqrt{1-x^2} \int_{x-c\varphi(x)/2n}^{x+c\varphi(x)/2n} |H_n''(f, w, t)|\, dt\, dx \\
&\leqslant c \int_{D_n} \int_{x-c\varphi(x)/2n}^{x+c\varphi(x)/2n} u(t)\sqrt{1-t^2}\, |H_n''(f, w, t)|\, dt\, dx \\
&\leqslant c \int_{D_n} u(t)\sqrt{1-t^2}\, |H_n''(f, w, t)| \left\{ \int_{|x-t|<c\varphi(x)/2n} dx \right\} dt \\
&\leqslant cn^{-1}\, \|(1-t^2)\, H_n''(f, w, t)\|_{u,1} = o(1). \qquad (2.7)
\end{aligned}$$

Lemma 2.1 follows by combining (2.5)–(2.7). ∎

Proof of Theorem 1.1. From (1.8) we deduce that

$$H_n(f, w, x) + \sum_{k=1}^{\infty} (H_{2^k n}(f, w, x) - H_{2^{k-1} n}(f, w, x))$$

converges a.e. on $[-1, 1]$; moreover, we have

$$f(x) = H_n(f, w, x) + \sum_{k=1}^{\infty} (H_{2^k n}(f, w, x) - H_{2^{k-1} n}(f, w, x)) \tag{2.8}$$

a.e on $[-1, 1]$. Thus (2.8) holds on $[-1, 1]\backslash A_1$ with $m(A_1) = 0$ and $A_1 \subset [-1, 1]$. On the other hand, from (2.1) we also have

$$0 = H'_n(f, w, x) + \sum_{k=1}^{\infty} (H'_{2^k n}(f, w, x) - H'_{2^{k-1} n}(f, w, x))$$

a.e. on $[-1, 1]$. That means that the subsequences $\{H'_{2^k n}(f, w, x)\}$ converges to 0 a.e. on $[-1, 1]$. There fore we obtain by applying Egoroff's Theorem that for any given $\varepsilon > 0$, there is a subset $A_2 \subset [-1, 1]$ with $m(A_2) < \varepsilon$ such that $H'_{2^k n}(f, w, x)$ converges to 0 uniformly on $[-1, 1]\backslash A_2$, hence together with (2.8) we conclude that $f'(x) = 0$ on $[-1, 1]\backslash (A_1 \cup A_2)$. On the other hand, $f' \in C[-1, 1]$ implies that $f'(x)$ is bounded with $\|f'\| \leqslant M$, where M is a some positive constant. Since ε is arbitrary, we have

$$\begin{aligned}
&\|(1-x^2) f'(x)\|_{u,p}^{p} \\
&\quad \leqslant c(p)(\|(1-x^2) f'(x)\|^{p}_{L^p_u([-1,1]\backslash(A_1 \cup A_2))} + \|(1-x^2) f'(x)\|^{p}_{L^p_u(A_1 \cup A_2)}) \\
&\quad \leqslant c(p)(o(1) + M\varepsilon);
\end{aligned}$$

this implies that $f'(x) = 0$ a.e. on $[-1, 1]$. Since $f'(x)$ is continuous, therefore we have $f(x) \equiv constant$.

Note that $H_n(1, w, x) \equiv 1$, so the converse statement is trivial. ∎

Proof of Theorem 1.2. We just prove (1.9). (1.10) can be proved by similar argument. Define a main-part K-functional:

$$\mathscr{K}_\varphi(f, t)_{u,p} := \sup_{0 < h \leqslant t} \inf_g \{\|f - g\|_{L^p_u(I_h)} + t\,\|\varphi g'\|_{L^p_u(I_h)} \mid g \in A.C.(I_h)\},$$

where $g \in A.C(I_h)$ means that g is absolutely continuous in I_h. Then (cf. [2, Theorem 6.1.1, p. 56]) we have

$$\begin{aligned}
&\|\Delta_{h\varphi} f\|_{L^p_u(I_h)} \leqslant c\mathscr{K}_\varphi(f, t)_{u,p} \\
&\quad \leqslant c \sup_{\tau \leqslant h} \inf_l (\|f(x) - H_{2^l}(f, w, x)\|_{u,p} + \tau \|\varphi(x)\, H'_{2^l}(f, w, x)\|_{u,p}),
\end{aligned} \tag{2.9}$$

where l is taken such that $2^l < 1/\tau \leqslant 2^{l+1}$.

We write $H'_{2^l}(f, w, x)=\sum_{k=0}^{l-1}(H_{2^{k+1}}(f, w, x)-H_{2^k}(f, w, x))'$; by the use of Bernstein–Markov inequality (2.2) we have

$$\|\varphi(x)\, H'_{2^l}(f, w, x)\|_{u,p}\leqslant \sum_{k=0}^{l-1}\|\varphi(x)\,(H_{2^{k+1}}(f, w, x)-H_{2^k}(f, w, x))'\|_{u,p}$$

$$\leqslant c\sum_{k=0}^{l-1}2^{k+1}\left(\frac{1}{2^k}\right)^{\gamma}\leqslant c(\gamma)(1+\tau^{\gamma-1}); \tag{2.10}$$

combining (2.9) (2.10) and (1.9) we have $\Omega_{\varphi}(f, t)_{u,p}=O(1)\, t^{\gamma}$. ∎

3. REMARKS

1. Recalling that $\|f\|_{u,p}\leqslant c\,\|f\|$ for $1\leqslant p<\infty$, hence, we have by applying Theorem 1.1 and Hölder inequality that

THEOREM 3.1 *Let $f\in C^1[-1,1]$, and $w(x)$ be a generalized Jacobi weight. Then*

$$\|H_n(f, w, x)-f(x)\|=o(n^{-1}) \tag{0.0}$$

if and only if $f(x)\equiv$ constant.

For the case $w(x)=v(x)$, (3.1) is the result of J. Szabados [8, Theorem 3, p. 405]. Thus, we generalize his result. For the further studying of the saturation in the maximum norm, see [9, Theorem 1, p. 463] and references therein. Combining Theorem 1.1 and Theorem 3.1, we conclude that the saturation order of mean convergence of Hermite–Fejér interpolation is exactly same as that of uniform metric.

2. From (1.5) we obtain that

$$\|H_n(f, v, x)-f(x)\|_{v,p}=O(n^{-1}) \tag{3.2}$$

for f satisfying $\omega_{\varphi}(f, t)\leqslant ct (t>0)$ and $1\leqslant p<\infty$. But how to characterize the saturation class of $H_n(f, w, x)$ in L^p space $(1\leqslant p<\infty)$ is still an open problem.

3. For the Quasi–Hermite–Fejér interpolation, there are similar results which hold, we omit the details.

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